

Renormalization of Gauge Theories and the Hopf Algebra of Diagrams

D.V. Prokhorenko *

February 1, 2008

Abstract

In 1999 A. Connes and D. Kreimer have discovered a Hopf algebra structure on the Feynman graphs of scalar field theory. They have found that the renormalization can be interpreted as a solving of some Riemann — Hilbert problem. In this work the generalization of their scheme to the case of nonabelian gauge theories is proposed. The action of the gauge group on the Hopf algebra of diagrams is defined and the proof that this action is in consistent with the Hopf algebra structure is given. The sketch of new proof of unitarity of S -matrix, based on the Hopf algebra approach is given.

*Institute of Spectroscopy, RAS 142190 Moscow Region, Troitsk

1 Introduction

The mathematical theory of renormalization (R -operation) was developed by N.N. Bogoliubov and O.S. Parasiuk [1]. K. Hepp has elaborated on their proofs [2].

In 1999, A. Connes and D. Kreimer [3, 4] have discovered a Hopf Algebra structure on the Feynman graphs in scalar field theory with φ^3 interaction. The Hopf algebras play an important role in the theory of quantum groups and other noncommutative theories. (About noncommutative field theory and its relation to p-adic analysis see [5, 6].)

In the Connes — Kreimer theory the Feynman amplitudes belongs to the group of characters of the Hopf algebra of diagrams. Denote by U a character corresponding to the set of nonrenormalized amplitudes. Denote by R the character corresponding to the set of renormalized amplitudes, and denote by C the character corresponding to the counterterms. The following identity holds:

$$R = C \star U. \tag{1}$$

Here, the star denotes the group operation in the group of characters.

Denote by $U(d)$ the dimensionally regularized Feynmann amplitude (d is a parameter of dimensional regularization). $U(d)$ is holomorphic in a small neighborhood of the point $d = 6$. We can consider $U(d)$ as a data for the Riemann — Hilbert problem [7] on the group of characters of Hopf algebra of diagrams. A. Connes and D. Kreimer have proved that this problem has an unique solution and the positive and negative parts of The Birkhoff de-

composition define renormalized amplitudes and counterterms (if we use the minimal subtraction scheme). About future generalization of this scheme see [8, 9, 10].

In [11] the generalization of this scheme to the case of quantum electrodynamics is given. In gauge theories it is necessary to prove that the renormalized Feynman amplitudes are gauge invariant. In quantum electrodynamics the condition of gauge invariance is expressed in terms of the Ward identities and in nonabelian gauge theories in terms of the Slavnov — Taylor identities.

Thus, an interesting problem is the problem of definition the action of gauge group on the Hopf algebra of Feynman graphs such that this action do not destroy the structure of Hopf algebra.

We solve this problem in the present paper.

Another the Hopf algebra description of renormalization theory of non-abelian gauge fields was proposed in [12].

The paper composed as follows. In section 2 we recall the basic concept of Hopf algebras. In section 3 we define the algebra of Feynman graphs (so-called Connes — Kreimer algebra) and prove that this algebra has an essential structure of Hopf algebras (so-called generalized Connes — Kreimer theorem).

In section 4 we recall the basic notion of gauge theories. In section 5 we recall the continual integral method for quantization gauge fields. In section 6 we derive the Slavnov — Taylor identities. Note that the usual Slavnov — Taylor identities are nonlinear, but our identities are linear. In section 7 we

derive the Slavnov — Taylor identities for individual diagrams. In section 9 we define the action of the gauge group on the Hopf algebra of diagrams and prove our main results which state that the action of gauge group do not destroy the Hopf Algebra Structure. In section 10 we show how to apply our results to the proof that physical observable quantities do not depend on the special chose of gauge conditions.

2 Hopf algebras

Definition. Coalgebra is a triple (C, Δ, ε) , where C is a linear space over the field \mathbf{k} ; $\Delta : C \rightarrow C \otimes C$, $\varepsilon : C \rightarrow \mathbf{k}$ are linear maps satisfying the following axioms:

A)

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \quad (2)$$

B) The following map:

$$(id \otimes \varepsilon) \circ \Delta : C \rightarrow C \otimes \mathbf{k} \cong C, \quad (3)$$

$$(\varepsilon \otimes id) \circ \Delta : C \rightarrow \mathbf{k} \otimes C \cong C \quad (4)$$

are identical. The map Δ is called a coproduct, and ε is called a counity. The property A) is called a coassociativity.

Definition. Coalgebra (A, Δ, ε) is a bialgebra if A is an algebra and the comultiplication and counit are homomorphism of algebras:

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1, \quad (5)$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(1) = 1. \quad (6)$$

Sweedler notation. Let (C, Δ, ε) be a coalgebra and let x be an element of C . $\Delta(x)$ have the following form

$$\Delta(x) = \sum_i x'_i \otimes x''_i \quad (7)$$

for some $x', x'' \in C$. This sum can be formally rewritten as follows

$$\Delta(x) = \sum_{(x)} x' \otimes x''. \quad (8)$$

This notations are called the Sweedler notation. In these terms the coassociativity axiom can be rewritten as follows

$$\sum_{(x)} \left(\sum_{(x')} (x')' \otimes (x')'' \right) \otimes x'' = \sum_{(x)} x' \otimes \left(\sum_{(x'')} (x'')' \otimes (x'')'' \right). \quad (9)$$

In Sweedler notation booth sides of these expressions can be rewritten in the form

$$\sum_{(x)} x' \otimes x'' \otimes x'''. \quad (10)$$

Definition. Let (C, Δ, ε) be a coalgebra, A be an algebra. Let f, g be linear maps $C \rightarrow A$; $f, g : C \rightarrow A$. By definition the convolution $f \star g$ of the maps f and g is the following map:

$$\mu \circ (f \otimes g) \circ \Delta : C \rightarrow A. \quad (11)$$

Here μ is an multiplication in A . $\mu : a \otimes b \mapsto ab$.

Definition. Let (A, Δ, ε) be a bialgebra. The antipode map S in this bialgebra is a linear map $A \rightarrow A$ such that

$$S \star id = id \star S = \eta \circ \varepsilon. \quad (12)$$

Here η is a homomorphism $\mathbf{k} \rightarrow A$, $x \mapsto \mathbf{1}x$ and $\mathbf{1}$ is a unit in A .

Definition. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebra over the field \mathbf{k} . Character χ on A is an homomorphism $A \rightarrow \mathbf{k}$.

Denote by G the set of all characters. The product of two characters χ and ρ is their convolution $\chi \star \rho$. One can check that $\chi \star \rho$ is an character. The convolution is associative. This fact follows from the coassociativity of Δ . There exists an identity ε in G . Indeed

$$(\varepsilon \star \chi)(x) = \sum_{(x)} \varepsilon(x') \chi(x'') = \sum_{(x)} \chi(\varepsilon(x') x'') = \chi\left(\sum_{(x)} \varepsilon(x') x''\right) = \chi(x). \quad (13)$$

Thus we have proved that ε is a right identity. Similarly one can prove that ε is a left identity. For each $\chi \in G$ there exists an inverse $\chi^{-1} = \chi \circ S$. Indeed

$$\begin{aligned} \chi \star (\chi \circ S)(x) &= \sum_{(x)} \chi(x') \chi(S(x'')) = \sum_{(x)} \chi(x' S(x'')) \\ &= \chi(\eta \circ \varepsilon(x)) = \chi(\mathbf{1}) \varepsilon(x) = \varepsilon(x). \end{aligned} \quad (14)$$

Similarly one can prove that $\chi \circ S$ is a left inverse of χ . Therefore the following theorem holds.

Theorem 1. *The set of all characters of a Hopf algebra is a group with respect the convolution as a group operation.*

Example. Let us consider the algebra H of all polynomial functions on $SL(2, \mathbb{C})$ with respect the pointwise multiplication. Then it is a Hopf algebra if we put:

$$\begin{aligned} (\Delta F)(g_1, g_2) &= F(g_1 g_2), \\ \varepsilon(F) &= F(e), \\ (S(F))(g) &= F(g^{-1}). \end{aligned} \quad (15)$$

Here g_1, g_2, g are elements of $SL(2, \mathbb{C})$, F is a polynomial function on $SL(2, \mathbb{C})$, e is the identity in $SL(2, \mathbb{C})$.

The group of characters G of H is isomorphic to $SL(2, \mathbb{C})$. This isomorphism to each element g of $SL(2, \mathbb{C})$ assigns a character χ_g , defined as

$$\chi_g(F) = F(g), \quad F \in H. \quad (16)$$

Definition. Let $C_1 = (A_1, \Delta_1, \varepsilon_1)$ and $C_2 = (A_2, \Delta_2, \varepsilon_2)$ be coalgebras. A homomorphism from C_1 to C_2 is a linear map $f : A_1 \rightarrow A_2$ such that

$$\Delta_2 \circ f = (f \otimes f) \circ \Delta_1, \quad (17)$$

$$\varepsilon_2 \circ f = \varepsilon_1. \quad (18)$$

Definition. Let $H_1 = (A_1, \Delta_1, \varepsilon_1, S_1)$ and $H_2 = (A_2, \Delta_2, \varepsilon_2, S_2)$ be Hopf algebras. The homomorphism $f : A_1 \rightarrow A_2$ is a Hopf algebra homomorphism $f : H_1 \rightarrow H_2$ if f is a coalgebra homomorphism $f : C_i \rightarrow C_i$, where $C_i = (A_i, \Delta_i, \varepsilon_i)$ ($i = 1, 2$), and

$$S_2 \circ f = f \circ S_1. \quad (19)$$

As usual in the case of Hopf algebra we can define the composition of homomorphisms and define the monomorphisms, epimorphisms etc.

Definition. Let $H = (A, \Delta, \varepsilon, S)$ be a Hopf algebra. A derivation δ of the Hopf algebra H is a derivation of A such that

$$\Delta \circ \delta = (id \otimes \delta + \delta \otimes id) \circ \Delta, \quad (20)$$

$$\varepsilon \circ \delta = 0, \quad (21)$$

$$S \circ \delta = \delta \circ S. \quad (22)$$

Remark. We can think about the derivatives as about the infinitesimal automorphism.

3 Feynman Diagrams

Let us define the Feynman diagrams. Suppose that the theory describes N fields Φ_a^α , where $a = 1, \dots, N$ is an index numerating different fields, α is an index, numerating different components of fields. (This index may be spinor, vector, group etc.) $\alpha = 1, \dots, \alpha_a$. For each field corresponding to the index a we assign its index space $\mathbb{Z}_a := \mathbb{C}^{\alpha_a}$ ($\mathbb{Z}_a := \mathbb{R}^{\alpha_a}$).

Definition. A Feynman graph is a triple $\Phi = (V, \{R_a\}_{a=1}^N, f)$, where V is a finite set, called a set of vertices, and $\forall a = 1, \dots, N$ R_a is a finite set, called a set of lines for the particles of type a . Put by definition $R = \bigcup_{a=1}^N R_a$. f is a map $f : R \rightarrow V \times V \cup V \times \{+, -\}$

Definition. Let $r \in R$ be a line $r \in f^{-1}(V \times V)$ or equivalently $f(r) = (v_1, v_2)$ for some vertices v_1 and v_2 . We say that the line r comes into the vertex v_1 and comes from the vertex v_2 . We say also that the vertices v_1 and v_2 are connected by the line r .

Let r be a line such that $f(r) = (v, +)$. We say that the line r is an external line coming from the vertex v . We also say that the line r comes from the Feynman graph G .

Let r be a line such that $f(r) = (v, -)$. We say that the line r is an external line coming into the vertex v . We also say that the line r comes into the Feynman graph G .

Definition. The Feynman graph Φ is called connected if for two any vertexes v, v' there exists a sequence of vertexes $v = v_0, v_1, \dots, v_n = v'$ such that $\forall i = 0, \dots, n-1$ the vertexes v_i and v_{i+1} are connected by some line.

Definition. A Feynman graph Φ is called one particle irreducible if it is connected and can not be disconnected by removing a single line.

Let Φ be a Feynman graph. Let v be a vertex of Φ . We let $R^{\rightarrow v}$ be a set of all lines coming into the vertex v , and $R^{\leftarrow v}$ be a set of all lines coming from the vertex v . Let:

$$\mathbb{Z}_v = \left\{ \bigotimes_{r \in R^{\leftarrow v}} \mathbb{Z}_{a_r} \right\} \bigotimes \left\{ \bigotimes_{r \in R^{\rightarrow v}} \mathbb{Z}_{a_r}^* \right\}. \quad (23)$$

Here a_r is a type of particle, corresponding to the line r . V^* is a dual of the space V .

Definition. The space \mathbb{Z}_v is called an index space of the vertex v .

Definition. Let Φ be a Feynman graph, and v be a vertex of Φ . The space S_v is a space of all linear combinations of the function of the form

$$\delta\left(\sum_{r \rightarrow v} p_r - p\right) f(p_r). \quad (24)$$

Here $f(p_r)$ is an arbitrary polynomial of variables $\{p_r | r \in R^{\leftarrow v} \cup R^{\rightarrow v}\}$ whose range is \mathbb{Z}_v .

Definition. The Feynman diagram is a pair $\Gamma = (\Phi, \varphi)$, where $\Phi = (V, \{R_a\}_{a=1}^N, f)$ is a Feynman graph and φ , is a map which assigns to each vertex $v \in V$ an element $\varphi(v)$ of S_v .

We will write below $\Phi_\Gamma, \varphi_\Gamma$, to point out that the Feynman graph Φ and the function φ corresponds to the diagram Γ .

Definition. Let $\Gamma = (\Phi, \varphi)$ be a diagram

$$\Phi = (V, \{R_a\}_{a=1}^N, f)$$

and I be a set of all its external lines. Let L_Γ be a set of all maps $I \rightarrow \mathbb{R}^4$, $i \mapsto p(i)$. L_Γ is called a space of external particle momenta.

Let $\Gamma = (\Phi, \varphi)$ be a Feynman diagram. Let $R^{\rightarrow\Gamma}$ be a set of all external lines of Φ coming into Φ . Let $R^{\leftarrow\Gamma}$ be a set of external lines of Φ coming from Φ . Let

$$\mathbb{Z}_\Gamma = \left\{ \bigotimes_{r \in R^{\leftarrow\Gamma}} \mathbb{Z}_{a_r} \right\} \bigotimes \left\{ \bigotimes_{r \in R^{\rightarrow\Gamma}} \mathbb{Z}_{a_r}^* \right\}. \quad (25)$$

Here a_r is a type of particle corresponding to r , and V^* is a dual space of the space V .

Definition. The space \mathbb{Z}_Γ is called an index space of the diagram Γ .

Definition. S_Γ is a space of all linear combination of the functions of the form

$$\delta\left(\sum_{r \in R^{\rightarrow\Gamma} \cup R^{\leftarrow\Gamma}} p_r - p\right) f(p_r), \quad (26)$$

Here $f(p)$ is a polynomial map from L_Γ to \mathbb{Z}_Γ .

Definition. Let S_Γ' be a algebraic dual of the space S_Γ . S_Γ' is called a space of external structure of Γ .

Definition. Let \mathcal{H} be a commutative unital algebra generated by the pairs (Γ, σ) (Γ is one particle irreducible diagram, $\sigma \in S_\Gamma'$) with the following

relations

$$\begin{aligned}(\Gamma, \lambda\sigma' + \mu\sigma'') &= \lambda(\Gamma, \sigma') + \mu(\Gamma, \sigma''), \\ (\lambda\Gamma' + \mu\Gamma'', \sigma) &= \lambda(\Gamma', \sigma') + \mu(\Gamma'', \sigma).\end{aligned}$$

Here Γ' , Γ'' and $\lambda\Gamma' + \mu\Gamma''$ are the diagrams such that

$$\Phi_{\Gamma'} = \Phi_{\Gamma''} = \Phi_{\lambda\Gamma' + \mu\Gamma''} \quad (27)$$

and there exists a vertex v_0 of $\Phi_{\Gamma'}$ such that

$$\begin{aligned}\varphi_{\Gamma'}(v) &= \varphi_{\Gamma''}(v) = \varphi_{\lambda\Gamma' + \mu\Gamma''}(v) \text{ if } v \neq v_0 \\ \varphi_{\lambda\Gamma' + \mu\Gamma''}(v_0) &= \lambda\varphi_{\Gamma'}(v_0) + \mu\varphi_{\Gamma''}(v_0).\end{aligned} \quad (28)$$

\mathcal{H} is called an algebra of Feynman diagrams.

Let us give some notation necessary to give a definition of coproduct on the algebra of Feynman diagrams.

Let $B_\Gamma = \{l_\Gamma^\alpha\}$, $\alpha \in A_\Gamma$ be an arbitrary Hamel basis of a space S_Γ^Ω . Denote by $B'_\Gamma = \{l_\Gamma^{\alpha'}\}$ the dual basis of $B_\Gamma = \{l_\Gamma^\alpha\}$.

Definition. Let $\Gamma = (\Phi, \varphi)$ be a one particle irreducible Feynman diagram, where $\Phi = (V, \{R_a\}_{a=1}^N, f)$. Let V' be a subset of V . Let \tilde{R}'_a be a subset of R_a for each $a = 1 \dots N$, such that $\forall r \in \tilde{R}'_a$ there exists vertices v_1 and v_2 from V' connected by r .

Let \tilde{R}''_a be a subset of $(R_a \setminus \tilde{R}'_a) \times \{+, -\}$, $\tilde{R}''_a := \tilde{R}''_a{}^+ \cup \tilde{R}''_a{}^-$. Here $\tilde{R}''_a{}^+$ is a set of all pairs $(r, +)$ such that $r \in R_a \setminus \tilde{R}'_a$ and r comes from V' , $\tilde{R}''_a{}^-$ is

a set of all pairs $(r, -)$ such that $r \in R_a \setminus \tilde{R}'_a$ and r comes into V' . Put by definition $R'_a = \tilde{R}'_a \cup \tilde{R}''_a$.

Let $\Phi_\gamma = (V', \{R'_a\}_{a=1}^N, f')$ be a Feynman Graph, where V' , R'_a are just defined and $f'(r) := f(r)$, if $r \in \tilde{R}'_a$, $f'((r, +)) = (v, +)$ if $(r, +) \in \tilde{R}''^{++}_a$ and $f(r) = (v', v)$ or $f(r) = (v, +)$; $f'((r, -)) = (v, -)$, if $(r, -) \in \tilde{R}''^{-+}_a$ and $f(r) = (v, v')$ or $f(r) = (v, -)$. Let $\gamma := (\Phi_\gamma, \varphi_\gamma)$, where φ_γ is a restriction of φ_Γ to V' . If Φ_γ is one particle irreducible diagram γ is called an one particle irreducible subdiagram of Γ .

Definition. Let $\gamma = \{\gamma_i | i = 1, \dots, n\}$ be a set of one particle irreducible subdiagrams of Γ such that $V_{\gamma_i} \cap V_{\gamma_j} \neq \emptyset \ \forall i \neq j$. We say that γ is a subdiagram of Γ . $\forall i = 1, \dots, n$ γ_i is called a connected component of γ . Let $M = \{1, \dots, n\}$. The elements of M numerate the connected components of M . Let α be a map which to each element of M assigns the element $\alpha(i)$ of A_{γ_i} . α is called a multi index. Let γ' be a subdiagram of $\Gamma = (\Phi, \varphi)$ and α be a multi index. We assign to the pair (γ', α) an element $\gamma_\alpha := \prod_{i \in M} (\gamma_i, l_{\gamma_i}^{\alpha(i)'})$ of \mathcal{H} .

The quotient diagram Γ/γ_α as a graph is obtained by replacing each of the connected component γ_i of γ by the corresponding vertex v_i . For each $i \in M$ we can identify S_{γ_i} with S_{v_i} . We put by definition $\varphi_{\Gamma/\gamma_\alpha}(v) = \varphi(v)$ if $v \neq v_i \ \forall i \in M$ and

$$\varphi_{\Gamma/\gamma_\alpha}(v_i) = l_{\gamma_i}^{\alpha(i)}.$$

Definition. Comultiplication Δ is a homomorphism $\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$, defined

on generators as follows:

$$\Delta((\Gamma, \sigma)) = (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma), \quad (29)$$

(see. [3, 4].)

Remark. In the previous formula \subset means the strong inclusion. The sum is over all nonempty subdiagrams $\gamma \subset \Gamma$ and multiindices α .

Theorem 2. *The homomorphism Δ is well defined and do not depend of a special chose of a basis B_Γ of S_Γ .*

Proof. It is evidence.

Theorem 2. (The generalized Connes — Kreimer theorem.)
Homomorphism Δ is coassociative. Moreover we can find a counit ε and an antipode S such that $(\mathcal{H}, \Delta, \varepsilon, S)$ is a Hopf algebra.

Proof. Let Γ be a Feynman diagram and $\gamma_\alpha, \gamma_\beta$ are subdiagrams of Γ such that $\gamma_\alpha \subset \gamma_\beta$. We can define a quotient diagram $\gamma_\beta/\gamma_\alpha$ by the evident way.

Let us show that Δ is coassociative. We have:

$$\Delta((\Gamma, \sigma)) = (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma), \quad (30)$$

$$\begin{aligned} (\Delta \otimes id) \circ \Delta((\Gamma, \sigma)) &= (\Gamma, \sigma) \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes (\Gamma, \sigma) \\ &\quad + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) \otimes \mathbf{1} \\ &\quad + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \Delta(\gamma_\alpha) \otimes (\Gamma/\gamma_\alpha, \sigma). \end{aligned} \quad (31)$$

$$(\Delta \otimes id) \circ \Delta((\Gamma, \sigma)) = (\Gamma, \sigma) \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes (\Gamma, \sigma)$$

$$\begin{aligned}
& + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) \otimes \mathbf{1} + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes \mathbf{1} \otimes (\Gamma/\gamma_\alpha, \sigma) \\
& + \mathbf{1} \otimes \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) + \sum_{\emptyset \subset \gamma_\beta \subset \gamma_\alpha \subset \Gamma} \gamma_\beta \otimes \gamma_\alpha/\gamma_\beta \otimes (\Gamma/\gamma_\alpha, \sigma). \quad (32)
\end{aligned}$$

From other hand:

$$\begin{aligned}
& (id \otimes \Delta) \circ \Delta((\Gamma, \sigma)) \\
& = (id \otimes \Delta) \{ (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) \} \\
& = (\Gamma, \sigma) \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes (\Gamma, \sigma) \\
& \quad + \mathbf{1} \otimes \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) \\
& + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma) \otimes \mathbf{1} + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} \gamma_\alpha \otimes \mathbf{1} \otimes (\Gamma/\gamma_\alpha, \sigma) \\
& \quad + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma; \emptyset \subset \gamma_\beta \subset \Gamma/\gamma_\alpha} \gamma_\alpha \otimes \gamma_\beta \otimes ((\Gamma/\gamma_\alpha)/\gamma_\beta, \sigma) \quad (33)
\end{aligned}$$

To conclude the prove of the theorem it is enough to prove the coincidence of the last terms of (32) and (33). In other words it is enough to prove the following equality

$$\begin{aligned}
& \sum_{\emptyset \subset \gamma_\beta \subset \gamma_\alpha \subset \Gamma} \gamma_\beta \otimes \gamma_\alpha/\gamma_\beta \otimes (\Gamma/\gamma_\alpha, \sigma) \\
& = \sum_{\emptyset \subset \gamma_\gamma \subset \Gamma; \emptyset \subset \gamma_\delta \subset \Gamma/\gamma_\gamma} \gamma_\gamma \otimes \gamma_\delta \otimes ((\Gamma/\gamma_\gamma)/\gamma_\delta, \sigma) \quad (34)
\end{aligned}$$

To each term of left hand side of (36)

$$\gamma_\beta \otimes \gamma_\alpha/\gamma_\beta \otimes (\Gamma/\gamma_\alpha, \sigma) \quad (35)$$

assign the following term of the right hand side of (36)

$$\gamma_\gamma \otimes \gamma_\delta \otimes ((\Gamma/\gamma_\gamma)/\gamma_\delta, \sigma), \quad (36)$$

where $\gamma = \beta$, $\gamma_\delta = \gamma_\alpha/\gamma_\beta$. It is evidence that this map is a bijection and $\Gamma/\gamma_\alpha = (\Gamma/\gamma_\gamma)/\gamma_\delta$. So the equality (36) holds. The coassociativity of Δ is proved.

It is easy to see that the homomorphism $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$ defined by $\varepsilon((\Gamma, \sigma)) = 0$, if $\Gamma \neq \emptyset$, $\varepsilon(\mathbf{1}) = 1$ is a counit in \mathcal{H}

Let $\tilde{\mathcal{H}}$ be a linear subspace of \mathcal{H} spanned by the elements $\mathbf{1}$ and $\{(\Gamma, \sigma)\}$. Let us define the linear function $S : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$ by using the following recurrent relations

$$S((\Gamma, \sigma)) = -(\Gamma, \sigma) - \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} (\gamma_\alpha) S((\Gamma/\gamma_\alpha, \sigma)). \quad (37)$$

The order of the diagrams in the right hand side less than n if the order of Γ is equal to n .

Now let us extend S to a map $S : \mathcal{H} \rightarrow \mathcal{H}$ by the following rule

$$S((\Gamma_1, \sigma_1) \dots (\Gamma_n, \sigma_n)) = S((\Gamma_1, \sigma_1)) \dots S((\Gamma_n, \sigma_n)). \quad (38)$$

One can prove that just defined map $S : \mathcal{H} \rightarrow \mathcal{H}$ is an antipode in \mathcal{H} . The Theorem is proved.

Definition. Let Γ be an one particle irreducible Feynman diagram. Let C_Γ be a space of all \mathbb{Z}_Γ -valued distributions on L_Γ which are finite linear combinations of the distributions of the form

$$\delta\left(\sum_{r \in R \rightarrow \Gamma \cup R \leftarrow \Gamma} p_r - p\right) f(p_r). \quad (39)$$

Here $f(p_r)$ is an arbitrary \mathbb{Z}_Γ -valued smooth function with compact support on L_Γ . Let C'_Γ be an algebraic dual of C_Γ . Let M be a linear space spanned by the pairs (Γ, σ) , $\sigma \in C'_\Gamma$ with relation expressing the linearity of (Γ, σ)

by Γ and σ . One can prove that M is a comodule over \mathcal{H} if one define the comultiplication on \mathcal{H} by the formula (29).

4 The Yang — Mills action

Let G be a compact semisimple Lie Group, \mathfrak{g} be its Lie algebra and $\hat{\cdot}$ be its adjoint representation. It is possible to find a basis of \mathfrak{g} (a set of generators) $\{T^a\}$ such that

$$\langle T^a T^b \rangle \equiv \text{tr} \hat{T}^a \hat{T}^b = -2\delta^{ab}. \quad (40)$$

Definition. Gauge field is a \mathfrak{g} -valued one-form on \mathbb{R}^4 :

$$A = \sum_{\mu=1}^4 \sum_a A_\mu^a dx^\mu T^a. \quad (41)$$

The covariant derivative. Let Γ be a representation of G by complex $n \times n$ matrices acting in $V = \mathbb{C}^n$.

Definition. Let R be a trivial bundle over \mathbb{R}^4 with the fibre V .

Let A_μ be a gauge field. The covariant derivative ∇_μ is a map

$$\nabla_\mu : \Gamma(R) \rightarrow \Gamma(R) \quad (42)$$

of the form

$$\nabla_\mu \psi = \partial_\mu \psi - g\Gamma(A_\mu)\psi, \quad \psi \in \Gamma(R). \quad (43)$$

Here $\Gamma(R)$ is a space of global sections of R .

Curvature. Let A be a gauge field. Its curvature are defined as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + g[A_\mu, A_\nu]. \quad (44)$$

One can easily check that

$$[\nabla_\mu, \nabla_\nu] = g\Gamma(F_{\mu\nu}). \quad (45)$$

Gauge transformation. Let $\omega(x)$ be a smooth map from \mathbb{R}^4 to G . Gauge transformation is an automorphism of R defined as

$$\psi(x) \rightarrow \psi'(x) = \Gamma(\omega(x))\psi(x). \quad (46)$$

Under the gauge transformation $\omega(x)$ the field A transforms as follows

$$A \rightarrow A'_\mu = \omega A_\mu \omega^{-1} + (\partial_\mu \omega) \omega^{-1}. \quad (47)$$

This rule follows from the formula

$$\nabla'_\mu \Gamma(\omega(x))\psi(x) = \Gamma(\omega(x))\{\nabla_\mu \psi(x)\}, \text{ where} \quad (48)$$

$$\nabla'_\mu = \partial_\mu - \Gamma(A'_\mu). \quad (49)$$

The curvature F under gauge transformations transforms as follows

$$F \rightarrow F' = \omega F \omega^{-1}. \quad (50)$$

The Yang — Mills action. Let Γ^a be an element of $\mathfrak{g}^{\mathbb{C}}$ (complexification of \mathfrak{g}) such that $T^a = i\Gamma^a$, where $i = \sqrt{-1}$. We have

$$\text{tr}(\hat{\Gamma}^a, \hat{\Gamma}^b) = 2\delta^{ab}. \quad (51)$$

By definition

$$[\Gamma^a, \Gamma^b] = if^{abc}\Gamma^c. \quad (52)$$

One can rewrite the curvature $F = F^a T^a$ as follows

$$F_{\mu\nu}^a = \partial_\nu A_\mu^a - \partial_\mu A_\nu^a - g f^{abc} A_\mu^b A_\nu^c. \quad (53)$$

The pure Yang — Mills action by definition has the form

$$S_{YM}[A] = -\frac{1}{8} \int \langle F_{\mu\nu}, F_{\mu\nu} \rangle d^4x = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a d^4x. \quad (54)$$

The action for fermions has the form

$$S_F = \int \bar{\psi} (i\gamma_\mu \nabla_\mu + m) \psi d^4x. \quad (55)$$

Here γ_μ are the Euclidean Dirac matrices. The action for the fermion interacting with the gauge field has the form

$$S = S_{YM} + S_F. \quad (56)$$

The action S is an invariant under the gauge transformation if the fermions under the gauge transformation transform as follows

$$\begin{aligned} \psi &\rightarrow \psi' = \omega \psi, \\ \bar{\psi} &\rightarrow \bar{\psi}' = \bar{\psi} \omega^{-1}. \end{aligned} \quad (57)$$

5 Quantization of the Yang — Mills theory

Let us recall the quantization procedure of the Yang — Mills theory by using the continual integral method.

Let $G[A, \bar{\psi}, \psi]$ be a gauge invariant functional, i.e. $G[A, \bar{\psi}, \psi]$ satisfies

$$G[\omega A, {}^\omega \bar{\psi}, {}^\omega \psi] = G[A, \bar{\psi}, \psi], \quad (58)$$

where

$$\begin{aligned}
{}^\omega A &:= \omega A \omega^{-1} + (\partial_\mu \omega) \omega^{-1}, \\
{}^\omega \psi &:= \omega \psi, \\
{}^\omega \bar{\psi} &:= \bar{\psi} \omega^{-1}.
\end{aligned} \tag{59}$$

The expectation value of the functional $G[A, \bar{\psi}, \psi]$ by definition can be expressed through the continual integral as follows

$$\langle G[A, \bar{\psi}, \psi] \rangle = \mathcal{N}^{-1} \int D A D \bar{\psi} D \psi G[A, \bar{\psi}, \psi] e^{-S[A]}. \tag{60}$$

Here \mathcal{N} is a constant such that $\langle 1 \rangle = 1$.

This integral contains the integration over the gauge group. Our aim is to include the volume of the gauge group into the \mathcal{N} .

Let $\chi[A](x)$ be a \mathfrak{g} -valued function on \mathbb{R}^4 depending of A ($\chi[A](x) = i\chi^a[A](x)\Gamma^a$). $\chi[A](x)$ are called gauge functions. By definition the gauge surface is a set of all field configurations $(A, \bar{\psi}, \psi)$ such that $\chi[A](x) = 0 \ \forall x \in \mathbb{R}^4$. We suppose that the gauge conditions are nondegenerate i.e.

$$\det \left\| \frac{\delta \chi^a[{}^\omega A](x)}{\delta \omega^b(y)} \right\| \neq 0 \tag{61}$$

if A belongs to the gauge surface. Let $\Delta[A]$ be a gauge invariant functional such that

$$\Delta[A] \int D\omega \delta(\chi[{}^\omega A]) = 1. \tag{62}$$

We have

$$\Delta[A] = \det \left\| \frac{\delta \chi[{}^\omega A]}{\delta \omega} \right\| \tag{63}$$

if the field configuration $(A, \bar{\psi}, \psi)$ lies on the gauge surface. We have

$$\begin{aligned} & \langle G[A_\mu, \bar{\psi}, \psi] \rangle \\ &= \mathcal{N}^{-1} \int DAD\bar{\psi}D\psi \int D\omega \delta(\chi[\omega A]) \Delta[A] e^{-S[A, \bar{\psi}, \psi]} G[A, \bar{\psi}, \psi]. \end{aligned} \quad (64)$$

The functional G , the action S , the measure $DAD\bar{\psi}D\psi$ and the functional $\Delta[A]$ are gauge invariant, therefore after the changing variables

$$\bar{\psi}, \psi, A \rightarrow \omega^{-1} \bar{\psi}, \omega^{-1} \psi, \omega^{-1} A \quad (65)$$

we can rewrite the last formula as follows

$$\begin{aligned} & \langle G[A\bar{\psi}, \psi] \rangle \\ &= \mathcal{N}^{-1} \int D\omega \int DAD\bar{\psi}D\psi \delta(\chi[A]) \det \left\| \frac{\delta \chi[\omega A]}{\delta \omega} \right\| e^{-S[A, \bar{\psi}, \psi]} G[A, \bar{\psi}, \psi]. \end{aligned} \quad (66)$$

Now we can include the integral $\int D\omega$ into the multiplier \mathcal{N}^{-1} .

The Faddeev — Popov ghosts. By definition the Faddeev — Popov ghosts are two \mathfrak{g} -valued Grassman fields $c^a(x)$ and $\bar{c}^a(x)$. We have

$$\det \left\| \frac{\delta \chi[\omega A]}{\delta \omega} \right\| = \int D\bar{c}Dc e^{\int \bar{c}^a(y) \frac{\delta \chi^a[\omega A](x)}{\delta \omega^b(y)} c^b(x) dx dy}. \quad (67)$$

Now let us use a new gauge conditions $\chi^{a'}[A](x) = \chi^a[A](x) - f^a(x) = 0$ in (66) instead of $\chi^a[A](x) = 0$, where f^a is an arbitrary \mathfrak{g} -valued function and integrate both sides of (66) over f^a with a weigh $e^{-\frac{1}{2} \int f^a(x) f^a(x) dx}$. In result we have

$$\langle G[A, \bar{\psi}, \psi] \rangle = \mathcal{N}^{-1} \int DAD\bar{\psi}D\psi D\bar{c}Dc G[A, \bar{\psi}, \psi] e^{-\{S_{YM} + S_F + S_{FP} + S_{GF}\}}, \quad (68)$$

where

$$S_{FP} = - \int \bar{c}^a(y) \frac{\delta \chi^a[\omega A](x)}{\delta \omega^b(y)} c^b(x) dx dy \quad (69)$$

and

$$S_{GF} = \frac{1}{2} \int (\chi^a[A](x))^2. \quad (70)$$

If we use the Lorentz gauge condition $\partial_\mu A_\mu = 0$ then we have

$$S_{FP} = \int \partial_\mu \bar{c}^a(y) \nabla_\mu c^a = -\frac{1}{2} \int \langle \partial_\mu \bar{c}, \nabla_\mu c \rangle. \quad (71)$$

By definition, under the gauge transformation the ghosts transforms as follows

$$\begin{aligned} \bar{c} &\mapsto \bar{c}, \\ c &\mapsto \omega c \omega^{-1}. \end{aligned} \quad (72)$$

6 The Slavnov — Taylor identities

Here we derive the Slavnov — Taylor identities. Note that our Slavnov — Taylor identities are linear but the usual Slavnov — Taylor identities are nonlinear.

The Green functions. Let us use the Lorenz gauge conditions. The Green functions are defined as

$$\begin{aligned} &\langle A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots, \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_k) \rangle \\ &= \int D A D \bar{\psi} D \psi D \bar{c} D c e^{-S} A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots, \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_k). \end{aligned} \quad (73)$$

The generating functional for the Green functions are defined as

$$Z[J, \bar{\eta}, \eta] = \int D A D \bar{\psi} D \psi D \bar{c} D c e^{-S + \langle J, A \rangle + \langle \bar{\eta}, \psi \rangle + \langle \bar{\psi}, \eta \rangle}, \quad (74)$$

where

$$\begin{aligned}
\langle J, A \rangle &:= \int J_\mu^a A_\mu^a d^4x, \\
\langle \bar{\eta}, \psi \rangle &:= \int \bar{\eta} \psi d^4x, \\
\langle \bar{\psi}, \eta \rangle &:= \int \bar{\psi} \eta d^4x.
\end{aligned} \tag{75}$$

Now we can calculate the Green functions as the functional derivatives of $Z[J, \bar{\eta}, \eta]$.

The generating functional for the connected Green functions are defined as

$$F[J, \bar{\eta}, \eta] = \ln Z[J, \bar{\eta}, \eta]. \tag{76}$$

At last, the generating functional for the one particle irreducible Green functions are defined by using the Legendre transformation

$$-\Gamma[A, \bar{\psi}, \psi] = \langle J, A \rangle + \langle \bar{\eta}, \psi \rangle + \langle \bar{\psi}, \eta \rangle - F[J, \bar{\eta}, \eta], \tag{77}$$

where $J, \bar{\eta}, \eta$ satisfy the conditions

$$\begin{aligned}
A &= \frac{\delta}{\delta J} F[J, \bar{\eta}, \eta], \\
\bar{\psi} &= -\frac{\delta}{\delta \eta} F[J, \bar{\eta}, \eta], \\
\psi &= \frac{\delta}{\delta \bar{\eta}} F[J, \bar{\eta}, \eta].
\end{aligned} \tag{78}$$

The Slavnov — Taylor identities Let $\omega = 1 + \alpha$ be an infinitesimal gauge transformation. Let us compute the following expression:

$$\delta_\omega \Gamma[\omega A, {}^\omega \bar{\psi}, {}^\omega \psi] := \Gamma[\omega A, {}^\omega \bar{\psi}, {}^\omega \psi] - \Gamma[A, \bar{\psi}, \psi]. \tag{79}$$

We have

$$\begin{aligned}
-\delta_\omega \Gamma[\omega A, \omega \bar{\psi}, \omega \psi] &= \langle J, \delta_\omega A \rangle + \langle \delta_\omega \bar{\psi}, \eta \rangle + \langle \bar{\eta}, \delta_\omega \psi \rangle \\
&+ \langle \delta_\omega J, A \rangle + \langle \bar{\psi}, \delta_\omega \eta \rangle + \langle \delta_\omega \bar{\eta}, \psi \rangle - \delta_\omega F[J].
\end{aligned} \tag{80}$$

The conditions (78) implies that

$$-\delta_\omega \Gamma[\omega A, \omega \bar{\psi}, \omega \psi] = \langle J, \delta_\omega A \rangle + \langle \bar{\delta}_\omega \psi, \eta \rangle + \langle \bar{\eta}, \delta_\omega \psi \rangle. \tag{81}$$

From other hand we have:

$$1 = \frac{Z[J]}{Z[J]} = \frac{1}{Z[J]} \int DAD\bar{\psi}D\psi D\bar{c}Dce^{-S+\langle J,A \rangle + \langle \bar{\eta}, \psi \rangle + \langle \bar{\psi}, \eta \rangle}. \tag{82}$$

It follows from the gauge invariance of the measure that

$$\begin{aligned}
0 &= \frac{1}{Z[J]} \int DAD\bar{\psi}D\psi D\bar{c}Dce^{\{-S+\langle J,A \rangle + \langle \bar{\eta}, \psi \rangle + \langle \bar{\psi}, \eta \rangle\}} \\
&\quad \{\langle J, \delta_\omega A \rangle + \langle \bar{\eta}, \delta_\omega \psi \rangle + \langle \delta_\omega \bar{\psi}, \eta \rangle - \delta_\omega S\}.
\end{aligned} \tag{83}$$

Let us introduce the following notation

$$S_\omega[A, \bar{\psi}, \psi] := S[\omega A, \omega \bar{\psi}, \omega \psi]. \tag{84}$$

Let $Z_\omega[J, \bar{\eta}, \eta]$, $F_\omega[J, \bar{\eta}, \eta]$, $\Gamma_\omega[A, \bar{\psi}, \psi]$ be generating functionals corresponding to the action S_ω . It follows from (83) that

$$\langle \delta_\omega \omega A, J \rangle + \langle \bar{\eta}, \delta_\omega \psi \rangle + \langle \delta_\omega \bar{\psi}, \eta \rangle + \delta_\omega F_\omega[J, \bar{\eta}, \eta] = 0, \tag{85}$$

but

$$\delta_\omega \Gamma_\omega[A, \bar{\psi}, \psi] = +\delta_\omega F_\omega[J, \bar{\eta}, \eta]. \tag{86}$$

Therefore

$$\delta_\omega \Gamma[\omega A, \omega \bar{\psi}, \omega \psi] = \delta_\omega \Gamma_\omega[A, \bar{\psi}, \psi]. \quad (87)$$

These equations we call the Slavnov — Taylor identities.

7 The Feynman rules for the Yang — Mills theory

We use the Lorenz gauge condition. The action has the form

$$S = \int \left\{ \frac{1}{4} \text{tr} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} (\partial_\mu A^a)^2 + \partial_\mu \bar{c}^a (\partial_\mu c - g[A_\mu, c])^a + \bar{\psi} (i\gamma_\mu \nabla_\mu + m) \psi \right\} d^4x. \quad (88)$$

The quadratic part of the action has the form

$$S_2 = \int \left\{ \frac{1}{2} (\partial_\nu A_\mu^a)^2 + \partial_\mu \bar{c}^a \partial_\mu c^a + \bar{\psi} (i\gamma_\mu \partial_\mu + m) \psi \right\} d^4x. \quad (89)$$

Let us write the terms describing the interaction.

The four-gluon interaction is described by the following vertex

$$\begin{aligned} V_{4A} &= -\frac{g^2}{4} \int [A_\mu, A_\nu]^a [A_\mu, A_\nu]^a d^4x \\ &= -\frac{g^2}{4} \int f^{abe} f^{cde} A_\mu^a A_\nu^b A_\mu^c A_\nu^d d^4x. \end{aligned} \quad (90)$$

The three gluon interaction is described by the vertex

$$V_{3A} = \frac{g}{2} \int \langle \partial_\nu A_\mu, [A_\mu, A_\nu] \rangle d^4x$$

$$= g \int \partial_\nu A_\mu^a A_\mu^b A_\nu^c f^{abc} d^4x. \quad (91)$$

The gluon-ghosts interaction is described by

$$\begin{aligned} V_{A\bar{c}c} &= -\frac{g}{2} \int \langle \partial_\mu \bar{c}, [A_\mu, c] \rangle dx \\ &= -g \int \partial_\mu \bar{c}^a A_\mu^b c^c f^{abc} dx. \end{aligned} \quad (92)$$

The fermion-gluon interaction

$$V_{A\bar{\psi}\psi} = ig\bar{\psi}\gamma_\mu A_\mu\psi = -gA_\mu^a\bar{\psi}\gamma_\mu\Gamma^a\psi. \quad (93)$$

Let us introduce the following notation for the Fourier transformation $\tilde{f}(k)$:

$$f(x) = \int e^{ikx} \tilde{f}(k) dk. \quad (94)$$

We have the following expression for the free gauge propagator

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle_0 = \delta^{ab} \delta^{\mu\nu} \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{k^2} dk, \quad (95)$$

for the free ghost propagator

$$\langle \bar{c}^a(x) c^b(y) \rangle_0 = \delta^{ab} \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{k^2} dk, \quad (96)$$

and for the free fermion propagator

$$\langle \bar{\psi}(x) \psi(y) \rangle_0 = \delta^{ab} \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{-\gamma_\mu k_\mu + m} dk. \quad (97)$$

In Fourier representation we have

$$\langle \tilde{A}_\mu^a(k) \tilde{A}_\nu^b(k') \rangle_0 = \delta^{ab} \delta^{\mu\nu} \frac{1}{(2\pi)^4} \delta(k+k') \frac{1}{k^2}, \quad (98)$$

$$\langle \tilde{c}^a(k) \tilde{c}^b(k') \rangle_0 = \delta^{ab} \frac{1}{(2\pi)^4} \delta(k+k') \frac{1}{k^2}. \quad (99)$$

Our aim is to define the gauge transformation on the Hopf algebra of Feynman graphs. First of all we must prove the Slavnov — Taylor identities for individual diagrams.

8 The Slavnov — Taylor identities for individual diagrams

Definition. Let v be a vertex of the diagram Γ . Suppose that n gluon lines come into v , m' fermion lines come from v , m fermion lines comes into v , k ghost lines comes into v and k' ghost lines comes from v . Let

$$w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}). \quad (100)$$

be an element of S_v (vertex operator) in coordinate representation. We assign to each such operator the following expression (Vick monomial)

$$\begin{aligned} V = & \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i, \end{aligned} \quad (101)$$

V is also called the vertex operator.

Let $\omega = 1 + \alpha$ be an infinitesimal gauge transformation, where α is an \mathfrak{g} -valued distribution such that its Fourier transform is a finite linear

combination of δ -functions

$$\tilde{\alpha}(k) = \sum_{i=1}^n c_i \delta(k - k_i). \quad (102)$$

The gauge variation $\delta_\alpha V$ of V by definition is a new vertex operator:

$$\begin{aligned} \delta_\alpha V = & g \sum_{i=1}^n \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots [\alpha(x_i), A(x_1)] \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\ & - g \sum_{i=1}^m \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_i) \alpha(y_i) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\ & + g \sum_{i=1}^{m'} \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \alpha(z_i) \psi(z_i) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\ & + g \sum_{i=1}^{k'} \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots [\alpha(w_i), c(w_i)] \dots c(w_{k'}) \\ & \times \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i. \quad (103) \end{aligned}$$

It is easy to see that $\delta_\alpha V \in S_v$.

Example 1. The gauge variation of four-gluon vertex is equal to zero.

Example 2. The gauge variation of the three gluon vertex is equal to

$$\delta_\alpha V_{3A} = \frac{g^2}{2} \langle [\partial_\nu \alpha, A_\mu], [A_\mu, A_\nu] \rangle. \quad (104)$$

Example 3. The gauge variation of the vertex describing the gluon-fermion interaction is equal to zero.

Example 4. The gauge variation of vertex, describing the ghost-fermion interaction is equal to

$$\delta_\alpha V_{\bar{c}cA} = -\frac{g}{2} \langle \partial_\mu \bar{c}, [\alpha, [A_\mu, c]] \rangle. \quad (105)$$

Now we must define the ξ -insertion into the vertecies and propagators.

ξ -insertion into the four-gluon vertex is equal to zero.

ξ -insertion into the three-gluon vertex is equal to

$$-\delta_\omega V_{3A} = -\frac{g^2}{2} \langle [\partial_\nu \alpha, A_\mu], [A_\mu, A_\nu] \rangle. \quad (106)$$

Remark. ξ -insertion into three-gluon vertex is a minus gauge transformation of this vertex.

ξ -insertion into the ghost-gluon vertex is equal to zero.

ξ -insertion into the fermion-gluon vertex is equal to zero.

ξ -insertion into the gluon line. To obtain a ξ -insertion into the gluon line we must insert into this line the following two-photon vertex.

$$\begin{aligned} & \frac{g}{2} \langle \partial_\nu \partial_\mu \alpha, [A_\mu, A_\nu] \rangle + \frac{g}{2} \langle \partial_\nu A_\mu [\partial_\mu \alpha, A_\nu] \rangle + \frac{g}{2} \langle \partial_\nu A_\mu, [A_\mu, \partial_\nu \alpha] \rangle \\ &= -\frac{g}{2} \langle \square A_\mu [A_\mu, \alpha] \rangle - \frac{g}{2} \langle \partial_\mu A_\mu, \partial_\mu [\alpha, A_\mu] \rangle. \end{aligned} \quad (107)$$

ξ -insertion into the ghost line. To obtain a ξ -insertion into the ghost line one must insert into this line the following two-ghost vertex.

$$-\frac{g}{2}\langle\partial_\mu\bar{c}, [\partial_\mu\alpha, c]\rangle. \quad (108)$$

ξ -insertion into the fermion line. To obtain a ξ -insertion into the fermion line one must insert the following two-fermion vertex into this line.

$$ig\bar{\psi}(x)\gamma_\mu(\partial_\mu\alpha)\psi(x). \quad (109)$$

η -insertions. We will see below that the η -insertions comes from gauge variations of the action. Let $\omega = 1 + \alpha$ be an infinitesimal gauge transformation. The gauge transformation of the action is equal to

$$\begin{aligned} \delta_\alpha S &= \delta_\alpha S_{G.F.} + \delta_\alpha S_{F.P.}, \\ \delta_\alpha S_{G.F.} &= -\frac{1}{2}\langle\partial_\mu A_\mu, \square\alpha\rangle + \frac{g}{2}\langle\partial_\mu A_\mu, \partial_\mu[A_\mu, \alpha]\rangle, \\ \delta_\alpha S_{F.P.} &= -\frac{1}{2}\langle\partial_\mu\bar{c}, [\alpha, [\nabla_\mu, c]]\rangle. \end{aligned} \quad (110)$$

η - insertion into the gluon line. To obtain a η - insertion into the gluon line we must to insert the following two-gluon vertex into this line

$$\frac{g}{2}\langle\partial_\mu A_\mu, \partial_\mu[\alpha, A_\mu]\rangle. \quad (111)$$

Remark. Note that the sum of ξ and η insertions into into the gluon line is equal to

$$\frac{g}{2}\langle\square A_\mu, [\alpha, A_\mu]\rangle. \quad (112)$$

η - insertion into the ghost line. To obtain a η -insertion into the ghost line one must to insert the following two-ghost vertex into this line

$$-\frac{g}{2}\langle\partial_\mu\bar{c}, [\alpha, \partial_\mu c]\rangle. \quad (113)$$

Remark. One can easily see that the sum of ξ - and η -insertion into the ghost line is equal to

$$\frac{g}{2}\langle\Box\bar{c}, [\alpha, c]\rangle. \quad (114)$$

η - insertion into the fermion-gluon vertex is equal to zero.

η - insertion into the ghost-gluon vertex replace the vertex operator $-\frac{g}{2}\langle\partial_\mu\bar{c}, [A_\mu, c]\rangle$ by

$$\frac{g^2}{2}\langle\partial_\mu\bar{c}, [\alpha, [A_\mu, c]]\rangle. \quad (115)$$

Remark. Note that the η - insertion into this vertex is equal to minus its gauge variation.

The Feynman rule for generating functional $\Gamma[A, \bar{\psi}, \psi]$.

To obtain the contribution from all one particle irreducible n -vertex diagrams into $\Gamma[A, \bar{\psi}, \psi]$ one must draw n points, then one must to replace each of this point by one of the vertexes from previous list, then we must connect this points by lines. We get diagrams. Then we must to each line assign a propagator etc. It is necessary to note that we do not identify topologically equivalent diagrams. The formalization of this procedure is simple and omitted.

Theorem 4. *The Slavnov — Taylor identity for individual diagrams. Let G be a one particle irreducible diagram without external ghost lines. Let*

G_ξ and G_η be diagrams, obtained from G by doing ξ - and η - insertion into some line or vertex of the diagram G . Denote by $\Gamma_G[A, \bar{\psi}, \psi]$ the contribution into the generating functional, corresponding by G . We have

$$\sum_{\xi} \Gamma_{G_\xi}[A, \bar{\psi}, \psi] + \sum_{\eta} \Gamma_{G_\eta}[A, \bar{\psi}, \psi] + \delta_\omega \Gamma_G[\omega A \omega^{-1}, \bar{\psi} \omega^{-1}, \omega \psi] = 0. \quad (116)$$

Here the first sum is over all ξ -insertions into the diagram, and the second sum is over all η -insertions into Γ .

Proof. Let us consider the sum of ξ - and η - insertion into gluon line. We have shown that this sum is equal to $-g \square A_\mu^a[\alpha, A_\mu]^a$. Not that the free propagator $\langle A_\mu^a(x) A_\nu^a(y) \rangle_0$ is a fundamental solution of the Laplace equation

$$\square_x \langle A_\mu^a(x) A_\nu^a(y) \rangle_0 = -\delta(x - y) \delta^{a,b} \delta_{\mu,\nu}. \quad (117)$$

We see that ξ - and η - insertions into the gluon lines leads to the gauge transformation of photon shoots which are the ends of the line.

Similarly one can see that ξ - and η -insertions into the ghost line leads to the gauge transformation of shoots which are the ends of the line.

The term

$$\delta_\omega \Gamma_G[\omega A, \omega \bar{\psi}, \omega \psi] \quad (118)$$

leads to the gauge transformation of all shoots corresponding to all external lines.

Let us now consider the gluon — fermion vertex. We have seen that all ξ - and η -insertions into the lines leads to the gauge variation of all shoots of

this vertex, i.e. to the gauge variation of this vertex. The ξ - and η -insertions into this vertex are equal to zero. But this vertex is a gauge invariant. Therefore the sum of all gauge variations of all shoots of this vertex and ξ - and η -insertions into the vertex is equal to zero.

Now let us consider the three-gluon vertex. We have seen that all the ξ - and η -insertions into the lines leads to the gauge variation of all shoots of this vertex, i.e. leads to the gauge transformation of this vertex. But η -insertion into this vertex is equal to zero and ξ -insertion into this vertex is equal to minus gauge variation of this vertex. Therefore the sum of all gauge variations of all shoots of this vertex and ξ - and η - insertions into this vertex is equal to zero.

Similarly we can consider the gluon — ghost and four-gluon vertices.

Theorem is proved.

By definition the sum of all ξ - and η -insertions into the fixed vertex v is called the ζ -insertion into v . We have seen that ζ -insertion into each vertex coming from the action $S = S_{YM} + S_F + S_{FP} + S_{GF}$ is precisely a minus gauge variation of this vertex. We have proved the Slavnov — Taylor identity only for the diagrams coming from the action S . To define the gauge transformation on the algebra of diagrams we must consider the diagrams containing arbitrary vertexes. Therefore we define a ζ -insertion into an arbitrary vertex v as a minus gauge transformation of this vertex. The following theorem holds.

Theorem 5. (Generalized Slavnov — Taylor equality.) *For each one particle irreducible diagram G (with arbitrary vertices) the following*

identity holds:

$$\delta_\omega \Gamma_G[\omega A \omega^{-1}, \bar{\psi} \omega^{-1}, \omega \psi] + \sum_\zeta \Gamma_{G_\zeta}[A, \bar{\psi}, \psi] = 0. \quad (119)$$

Proof. The proof of this theorem is a copy of the prove of previous theorem.

Now let us show how to derive the Slavnov — Taylor identity from the Slavnov — Taylor identity for the individual diagrams. To simplicity we consider only the case of pure Yang — Mills theory.

Let us summarize the identities (116) over all one-particle irreducible diagrams. The sum over all η insertion is precisely a $-\delta_\omega \Gamma_\omega[A]$. The sum over all diagrams of $\delta_\omega \Gamma[\omega A \omega^{-1}]_G$ is equal to $\delta_\omega \Gamma[\omega A \omega^{-1}]$. Let us show that the sum over all ξ -insertions of $\Gamma[A]_{G_\xi}$ is equal to $\int \frac{\delta \Gamma[A]}{\delta A_\mu} \partial_\mu \alpha d^4 x$. If we prove this fact the statement will be proved because

$$\begin{aligned} \delta_\omega \Gamma[\omega A] &= \int \frac{\delta \Gamma[A]}{\delta A_\mu} (\partial_\mu \alpha - [A, \alpha]_\mu) \\ &= \delta_\omega \Gamma[\omega A \omega^{-1}] + \int \frac{\delta \Gamma[A]}{\delta A_\mu} \partial_\mu \alpha. \end{aligned} \quad (120)$$

We have the following representation for the generating functional

$$\Gamma[A] = \sum_n \frac{1}{n!} \sum_m \frac{1}{m!} \int \Gamma_n^m(x_1, \dots, x_m) A(x_1) \dots A(x_m) dx_1 \dots dx_m. \quad (121)$$

Here Γ_n^m is a sum of Feynman amplitude over all one particle irreducible diagrams with n vertices and m external lines (shoots). We suppose that the vertices and the external lines are not identical. Let us represent Γ_n^m as $\Gamma_n^m = \sum_{G_n} \Gamma_{G_n}^m$. Here the last sum is over all one particle irreducible diagrams

with n vertices and m external lines. We have

$$\begin{aligned} & \int \frac{\delta \Gamma[A]}{\delta A_\mu} \partial_\mu \alpha \\ &= \sum_n \frac{1}{n!} \sum_m \frac{1}{(m-1)!} \int \sum_{G_n} \Gamma_{G_n}^m(x_1, \dots, x_m) \partial \alpha(x_1) \dots A(x_m) dx_1 \dots dx_m, \end{aligned} \quad (122)$$

We can rewrite the last formula as follows

$$\begin{aligned} & \int \frac{\delta \Gamma[A]}{\delta A_\mu} \partial_\mu \alpha \\ &= \sum_n \frac{1}{(n)!} \sum_m \frac{1}{(m)!} \int \sum_{G_n} \Gamma_{G_n}^m(x_0, \dots, x_m) \partial \alpha(x_0) \dots A(x_m) dx_0 \dots dx_m. \end{aligned} \quad (123)$$

Here we begin numerate vertices and external lines from zero.

Let G_n^m be a one particle irreducible diagram with n vertices and m external lines. Let ξ be a ξ -insertion into some vertex or line. To each pair (G_n^m, ξ) assign a diagram $(G_n^m)_\xi$ by doing a ξ -insertion. One can easily show what we can rewrite the right hand side of (123) as follows

$$\sum_n \frac{1}{(n)!} \sum_m \frac{1}{(m)!} \int \sum_{(G_n^m, \xi)} \Gamma_{(G_n^m)_\xi}^m(x_1, \dots, x_m) A(x_1) \dots A(x_m) dx_0 \dots dx_m. \quad (124)$$

The right hand side of (124) is equal to $\sum_G \sum_\xi \Gamma_{G_\xi}[A]$. Therefore the Slavnov — Taylor identity is proved.

9 Gauge transformation on the Hopf algebra of diagrams

At first we must give some definition. Let Γ be an one particle irreducible diagram. Suppose that n gluon lines come into Γ , m fermion lines comes into

Γ and m' fermion lines come from Γ , k ghost lines comes into Γ and k' ghost lines comes from Γ . Let

$$f(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}). \quad (125)$$

be an element of S_Γ in coordinate representation. We assign to this element the following expression (vertex operator)

$$\begin{aligned} V_f = & \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i. \end{aligned} \quad (126)$$

Let α be a \mathfrak{g} -valued distribution on \mathbb{R}^4 which Fourier transform has the form

$$\alpha(\tilde{k}) = c\delta(k - k_0) \quad (127)$$

By definition the gauge variation of V_f is a new vertex operator

$$\begin{aligned} \delta_\alpha V_f = & g \sum_{i=1}^n \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots [\alpha(x_i), A(x_1)] \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\ & - g \sum_{i=1}^m \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\ & \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_i) \alpha(y_i) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\ & \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\ & + g \sum_{i=1}^{m'} \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \end{aligned}$$

$$\begin{aligned}
& \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \alpha(z_i) \psi(z_i) \dots \psi(z_{m'}) \\
& \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots c(w_{k'}) \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i \\
& + g \sum_{i=1}^{k'} \int w(x_1, \dots, x_n | y_1, \dots, y_m | z_1, \dots, z_{m'} | v_1, \dots, v_k | w_1, \dots, w_{k'}) \\
& \times A(x_1) \dots A(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_m) \psi(z_1) \dots \psi(z_{m'}) \\
& \times \bar{c}(v_1) \dots \bar{c}(v_k) c(w_1) \dots [\alpha(w_i), c(w_i)] \dots c(w_{k'}) \\
& \times \prod_{i=1}^n dx_i \prod_{i=1}^m dy_i \prod_{i=1}^{m'} dz_i \prod_{i=1}^k dv_i \prod_{i=1}^{k'} dw_i. \quad (128)
\end{aligned}$$

It is easy to see that this definition is well defined i.e. $\delta_\alpha V_f = V_{\delta_\alpha f}$ for some unique function $\delta_\alpha f \in S_\Gamma$. Let $\sigma \in S'_\Gamma$. By definition let $\delta_\alpha(\sigma)$ be an element of $\sigma \in S'_\Gamma$ such that

$$\langle \delta_\alpha(\sigma), f \rangle = \langle \sigma, \delta_\alpha(f) \rangle. \quad (129)$$

Here $\langle \sigma, f \rangle$ means the value of functional σ on g .

Definition. Let α be a \mathfrak{g} -valued distribution on \mathbb{R}^4 such that its Fourier transform has the form

$$\tilde{\alpha}(k) = \sum_{j=1}^n c_j \delta(p_j - p). \quad (130)$$

By definition the gauge transformation δ_α on \mathcal{H} is its derivative as an algebra defined on generators as follows

$$\delta_\alpha((\Gamma, \sigma)) = \delta'_\alpha((\Gamma, \sigma)) + \delta''_\alpha((\Gamma, \sigma)), \quad (131)$$

where we put

$$\delta'_\alpha((\Gamma, \sigma)) = \sum_{\zeta \in \Gamma} (\Gamma_\zeta, \sigma), \quad (132)$$

and

$$\delta''_\alpha((\Gamma, \sigma)) = (\Gamma, \delta_\alpha(\sigma)). \quad (133)$$

Theorem 6. *The gauge transformation is a derivative of \mathcal{H} , i.e.*

$$\Delta \circ \delta_\alpha = (\mathbf{1} \otimes \delta_\alpha + \delta_\alpha \otimes \mathbf{1}) \circ \Delta,$$

$$\varepsilon \circ \delta_\alpha = 0,$$

$$S \circ \delta_\alpha = \delta_\alpha \circ S.$$

Proof. We have

$$\begin{aligned} & \Delta \circ \delta_\alpha((\Gamma, \sigma)) \\ &= \Delta \circ \delta'_\alpha((\Gamma, \sigma)) + \Delta \circ \delta''_\alpha((\Gamma, \sigma)). \end{aligned} \quad (134)$$

it is evidence that

$$\Delta \circ \delta''_\alpha((\Gamma, \sigma)) = (\mathbf{1} \otimes \delta''_\alpha) \circ \Delta((\Gamma, \sigma)). \quad (135)$$

Therefore we must calculate:

$$\Delta \circ \delta'_\alpha((\Gamma, \sigma)) = \Delta\left(\sum_{\zeta} (\Gamma_\zeta, \sigma)\right). \quad (136)$$

We have

$$\begin{aligned} & \sum_{\zeta} \Delta((\Gamma_\zeta, \sigma)) \\ &= (\Gamma, \sigma) \otimes \mathbf{1} + \mathbf{1} \otimes (\Gamma, \sigma) + \sum_{\zeta} \sum_{\gamma_\alpha \subset \Gamma_\zeta} \gamma_\alpha \otimes (\Gamma_\zeta / \gamma_\alpha, \sigma). \end{aligned} \quad (137)$$

But the last sum is equal to

$$\begin{aligned}
& \sum_{\zeta} \sum_{\gamma_{\alpha} \subset \Gamma_{\zeta}} \gamma_{\alpha} \otimes (\Gamma_{\zeta}/\gamma_{\alpha}, \sigma) \\
&= \sum_{\gamma_{\alpha} \subset \Gamma} \sum_{\zeta \in \gamma} (\gamma_{\alpha})_{\zeta} \otimes (\Gamma/\gamma_{\alpha}, \sigma) \\
& \quad + \sum_{\gamma_{\alpha} \subset \Gamma} \sum'_{\zeta} \gamma_{\alpha} \otimes (\Gamma_{\zeta}/\gamma_{\alpha}, \sigma). \tag{138}
\end{aligned}$$

Here ' at the last sum means that all the ζ -insertions into the sum are the ζ -insertions into the vertices or lines of Γ which do not belong to γ .

Let us transform the first term in the right hand side of (138). We have

$$\begin{aligned}
& \sum_{\gamma_{\alpha} \subset \Gamma} \sum_{\zeta} (\gamma_{\alpha})_{\zeta} \otimes (\Gamma/\gamma_{\alpha}, \sigma) \\
&= \sum_{\gamma_{\alpha} \subset \Gamma} \sum_{\zeta} (\gamma_{\alpha})_{\zeta} \otimes (\Gamma/\gamma_{\alpha}, \sigma) \\
& \quad + \sum_{\gamma_{\alpha} \subset \Gamma} \delta''_{\alpha}(\gamma_{\alpha}) \otimes (\Gamma/\gamma_{\alpha}, \sigma) \\
& \quad - \sum_{\gamma_{\alpha} \subset \Gamma} \delta''_{\alpha}(\gamma_{\alpha}) \otimes (\Gamma/\gamma_{\alpha}, \sigma). \tag{139}
\end{aligned}$$

By definition of δ_{α} the sum of first and second terms in the right hand side of (139) is equal to

$$(\delta_{\alpha} \otimes \mathbf{1}) \sum_{\gamma_{\alpha} \subset \Gamma} \gamma_{\alpha} \otimes (\Gamma/\gamma_{\alpha}, \sigma). \tag{140}$$

The last term in right hand side is equal to

$$\sum_{\gamma_{\alpha} \subset \Gamma} \sum''_{\zeta} \gamma_{\alpha} \otimes ((\Gamma/\gamma_{\alpha})_{\zeta}, \sigma). \tag{141}$$

Here '' means that all ζ -insertion are made into the vertices of Γ/γ_{α} obtained by replacing of all connected components of γ by vertices. As result we have

$$\Delta \circ \delta_\alpha((\Gamma, \sigma)) = ((\delta_\alpha \otimes \mathbf{1}) + (\mathbf{1} \otimes \delta_\alpha)) \circ \Delta((\Gamma, \sigma)). \quad (142)$$

It follows from this fact that

$$\Delta \circ \delta_\alpha = (\mathbf{1} \otimes \delta_\alpha + \delta_\alpha \otimes \mathbf{1}) \circ \Delta. \quad (143)$$

Similarly one can prove that

$$\begin{aligned} \varepsilon \circ \delta_\alpha &= 0, \\ S \circ \delta_\alpha &= \delta_\alpha \circ S. \end{aligned}$$

The theorem is proved.

Remark. Below we will consider only characters U such that $U((\Gamma, l_\Gamma^{\alpha'})) \neq 0$, $l_\Gamma^{\alpha'} \in B'_\Gamma$ only for finite number of elements $\alpha \in B'_\Gamma$. For any two such characters U_1 and U_2 its product $U_1 \star U_2$ well defined.

Remark. Let \mathcal{G} be a linear space of all \mathfrak{g} -valued functions on \mathbb{R}^4 of the form

$$\sum_{i=1}^N a_i e^{ik_i x}. \quad (144)$$

\mathcal{G} is a Lie algebra with respect to the following Lie brackets

$$[\alpha_1, \alpha_2](x) = [\alpha_1(x), \alpha_2(x)]. \quad (145)$$

Theorem 7. *Gauge transformation δ is a homomorphism from \mathcal{G} to the Lie algebra of all derivatives of \mathcal{H} .*

Remark. We can define a gauge transformation δ_α on comodule M by using the formulas similar to (131, 132, 133). We find that δ_α is a derivative of comodule M i.e.

$$\Delta \circ \delta_\alpha(x) = (\mathbf{1} \otimes \delta_\alpha + \delta_\alpha \otimes \mathbf{1}) \circ \Delta(x). \quad (146)$$

Definition. Let α be a \mathfrak{g} -valued function on \mathbb{R}^4 of the form (144). We say that character U is gauge invariant if

$$\delta_\alpha^*(U) := U \circ \delta_\alpha = 0 \quad (147)$$

Remark. Let M' is an algebraically dual module of M over the group algebra of G . Dimensionally regularized Feynman amplitude define an element $m \in M'$. We say that $m \in M'$ is gauge invariant if $m \circ \delta_\alpha = 0 \ \forall \alpha$ of the form (144).

Theorem 8. *The element $m \in M'$ corresponding to dimensionally regularized Feynman amplitude is gauge invariant.*

Proof. This theorem follows from the Slavnov — Taylor identities for diagrams.

Theorem 9. *The set of all gauge invariant characters of G is a group.*

Proof. Let U_1 and U_2 be gauge invariant characters. We have:

$$\begin{aligned} U_1 \star U_2 \circ \delta_\alpha &= U_1 \otimes U_2 \circ \Delta \circ \delta_\alpha \\ &= (U_1 \otimes U_2) \circ ((\mathbf{1} \otimes \delta_\alpha) + (\delta_\alpha \otimes \mathbf{1})) \circ \Delta \\ &= (U_1 \circ \delta_\alpha) \star U_2 + U_1 \star (U_2 \circ \delta_\alpha) = 0. \end{aligned} \quad (148)$$

So the product of two gauge invariant character is a gauge invariant character. Let us prove that for each character U its inverse character U^{-1} is a gauge invariant. Indeed

$$U^{-1} \circ \delta_\alpha = U \circ S \circ \delta_\alpha = U \circ \delta_\alpha \circ S = 0. \quad (149)$$

Theorem is proved.

Definition. Character U is called gauge invariant up to degree n if $\delta^*(U)((\Gamma, \sigma)) = 0$ for all diagrams Γ which contain at most n vertices.

Remark. Let C be gauge invariant character up degree $n - 1$ and U be a character. One can prove that

$$\begin{aligned} & \{U(\bullet) + \sum_{\emptyset \subset \gamma_\alpha \subset \bullet} C(\gamma_\alpha)U(\bullet/\gamma_\alpha)\}\delta_\alpha((\Gamma, \sigma)) \\ &= \{\delta_\alpha^*U(\bullet) + \sum_{\emptyset \subset \gamma_\alpha \subset \bullet} C(\gamma_\alpha)(\delta_\alpha^*U)(\bullet/\gamma_\alpha)\}((\Gamma, \sigma)). \end{aligned} \quad (150)$$

For any diagram Γ which contain at most n vertices.

Definition. Let D be a open set in \mathbb{C} . The function $U_z, z \rightarrow U_z$ is called continuous, holomorphic, etc. if

- a) $\forall X \in \mathcal{H} U_z(X)$ is a continuous, holomorphic, ect. in D .
- b) For any diagram $\Gamma, l_\Gamma^{\alpha'} \in B_\Gamma U_z((\Gamma, \alpha')) \equiv 0$ in D for all elements $l_\Gamma^{\alpha'} \in B_\Gamma$ except some finite subset.

Riemann — Hilbert problem. Let U_z — be a character holomorphic, in some small punctured neighbourhood of zero $O \setminus \{0\}$. It is aimed to find two characters R_z and C_z , holomorphic in z in O and $\mathbb{C} \setminus \{0\}$ respectively such that the following identities holds

$$R_z = C_z \star U_z \quad (151)$$

in $\mathcal{O} \setminus \{0\}$ and $C_z \rightarrow \mathbf{1}$ if $z \rightarrow \infty$. The pair (R_z, C_z) is called the Birkhoff decomposition of U_z .

The uniqueness of solution follows from the Liouville theorem.

Theorem 10 (Connes — Kreimer). *The Riemann — Hilbert problem for group of characters has a solution.*

Proof. One can find the following explicit formulas for the solution of the problem:

$$C_z((\Gamma, \sigma)) = -\mathbf{T}(U_z((\Gamma, \sigma))) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} C_z((\gamma_\alpha, \sigma)) U_z((\Gamma/\gamma_\alpha, \sigma)), \quad (152)$$

$$R_z((\Gamma, \sigma)) = (1 - \mathbf{T})(U_z((\Gamma, \sigma))) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} C_z(\gamma_\alpha) U_z((\Gamma/\gamma_\alpha, \sigma)). \quad (153)$$

By definition an operator \mathbf{T} assigns to each Laurent series

$$\sum_{j=-n}^{\infty} a_j z^j \quad (154)$$

the following polynomial on z^{-1}

$$\sum_{j=-n}^{-1} a_j z^j. \quad (155)$$

The Riemann — Hilbert problem on M' . Let $m_z \in M'$ be an element of M' holomorphic in some punctured neighbourhood of zero $\mathcal{O} \setminus \{0\}$. This means that $\forall (\Gamma, \sigma) \in M$ $m_z((\Gamma, \sigma))$ holomorphic in $\mathcal{O} \setminus \{0\}$. It is aimed to find element $C_z \in G$ and $m_z^+ \in M'$ holomorphic in $\overline{\mathcal{C}} \setminus \{0\}$ and \mathcal{O} respectively such that in $\mathcal{O} \setminus \{0\}$ the following identities hold

$$m_z^+ = C_z \star m_z, \quad (156)$$

and

$$C_z(\infty) = \varepsilon. \quad (157)$$

Remark. If m_z corresponds to dimensionally regularized Feynman amplitudes then the existence of solution follows from the Bogoliubov — Parasiuk theorem.

Theorem 11. *If the solution of the Riemann — Hilbert problems (151, 156) exist and the data of these problems are gauge invariant then the elements of their Birkhoff decompositions (R_z, C_z) , $((m_z^+, C_z))$ are gauge invariant too.*

Proof. The proof follows from the fact that \mathbf{T} commutes with δ_α and from remark to theorem 1.

Remark. Let m_z be an element of M' corresponding to the set of dimensionally regularized Feynman amplitudes. Character m_z^+ corresponds to renormalized Feynman amplitudes and C_z corresponds to counterterms.

10 Independence of vacuum expectation value of gauge invariant functional of the chose of gauge condition

Let us show (on physical level of curiosity) that expectation value of gauge invariant functional does not depend of the chose of gauge condition.

Let us denote non-renormalized Green function as

$$\langle \dots A \dots \bar{c} \dots c \dots \rangle = \int DAD\bar{c}Dc e^{-S[A, \bar{c}, c]} \dots A \dots \bar{c} \dots c \dots \quad (158)$$

For simplicity we consider the case of pure Yang — Mills theory.

Denote renormalized Green functions as

$$\langle \dots A \dots \bar{c} \dots c \dots \rangle_R = \int DAD\bar{c}Dc \{e^{-S[A, \bar{c}, c]}\}_R \dots A \dots \bar{c} \dots c \dots \quad (159)$$

Let $F_R[J]$ be a generating functional for renormalized connected Green function, and $\Gamma_R[J]$ be its Legendre transform. One can prove that $\Gamma_R[A]$ is a generating functional for one particle irreducible renormalized Green functions.

Let $\omega = 1 + \alpha$ be an infinitezimal gauge transformation. Now let us use gauge condition $g[\omega A] = 0$ instead the gauge condition $g[A] = 0$. Let us denote the expectation value corresponding to the new gauge condition $g[\omega A] = 0$ by $\langle \rangle'$. We find that

$$\langle \dots A \dots \bar{c} \dots c \dots \rangle'_R = \int DAD\bar{c}Dc \{e^{-S[\omega A, {}^\omega \bar{c}, {}^\omega c]}\}_R \dots A \dots \bar{c} \dots c. \quad (160)$$

Recall that under the gauge transformation the ghosts transforms as follows

$$\begin{aligned} c &\mapsto \omega c \omega^{-1}, \\ \bar{c} &\mapsto \bar{c}. \end{aligned} \quad (161)$$

In (160) we must at first make the gauge transformation and then make the renormalization. The Legendre transformation assign to the connected

generating functional for green functions (160) the functional $(\Gamma_R)_\omega[A]$. It follows from the gauge invariance of renormalized Feynman amplitudes that

$$(\Gamma_R)_\omega[A] = \Gamma_R[\omega A]. \quad (162)$$

But inverse Legendre transformation assigns to $\Gamma_R[\omega A]$ the following set of Green functions

$$\langle \dots A \dots \bar{c} \dots c \dots \rangle_R'' = \int DAD\bar{c}Dc \{e_R^{-S}[\omega A, \omega \bar{c}, \omega c]\} \dots A \dots \bar{c} \dots c \dots \quad (163)$$

Here one must at first make renormalization of e^{-S} and then make the gauge transformation.

Let $F[A, \bar{c}, c]$ be an enough regular gauge invariant functional. Consider its expectation value for the new gauge condition $g[\omega A] = 0$. We have

$$\begin{aligned} \langle F[A, \bar{c}, c] \rangle' &= \int DAD\bar{c}Dc \{e^{-S}[\omega A, \omega \bar{c}, \omega c]\}_R F[A, \bar{c}, c] \\ &= \int DA^{\omega^{-1}} D\bar{c}Dc^{\omega^{-1}} \{e^{-S}[A, \bar{c}, c]\}_R F[\omega^{-1} \bar{c}, \omega^{-1} c] \\ &= \int DAD\bar{c}Dc \{e^{-S}\}_R [A, \bar{c}, c] F[A, \bar{c}, c] = \langle F[A, \bar{c}, c] \rangle. \end{aligned} \quad (164)$$

Therefore

$$\langle F[A, \bar{c}, c] \rangle' = \langle F[A, \bar{c}, c] \rangle. \quad (165)$$

The statement is proved. Note that we have proved the following statement.

Proposition. For any enough regular gauge invariant functional $F[A]$ the following identity holds:

$$\begin{aligned} \int DAD\bar{c}Dc \left\{ \int dx^4 \alpha(x) \frac{\delta e^{-S}[\omega A, \omega \bar{c}, \omega c]}{\delta \alpha(x)} \right\}_R F[A, \bar{c}, c], \\ \omega = 1 + \alpha. \end{aligned} \quad (166)$$

Now let $g'[A] = 0$ be a new gauge condition such that the difference $g'[A] - g[A]$ is infinitely small. Let A be a field configuration such that $g[A] = 0$. Let A' be a field configuration such that $g'[A'] = 0$. Suppose that A and A' belong to the same class of gauge equivalent field configuration. There exists an infinitely small function $\alpha[A](x)$ of x which is gauge invariant functional of A such that

$$A' = A + \nabla_A \alpha. \quad (167)$$

This statement follows from

$$\det \left\| \frac{\delta g[\omega A]}{\delta \omega} \right\| \neq 0. \quad (168)$$

There exist enough many functions in this class of the form

$$\alpha[A](x) = \sum_{i=1}^n f_i(x) G_i[A], \quad (169)$$

where $f_i(x)$ are well \mathfrak{g} -valued functions on \mathbb{R}^4 and $G_i[A]$ are enough regular functionals of A . Let us find the variation $\delta \langle F[A] \rangle$ corresponding to the variation $\delta g[A] = g'[A] - g[A]$ of gauge function. We have

$$\begin{aligned} & \delta \langle F[A] \rangle \\ & \int DAD\bar{c}Dc \left\{ \int d^4x \frac{\delta e^{-S[\omega A, \omega \bar{c}, \omega c]}}{\delta \alpha(x)} \sum_{i=1}^n f_i(x) G_i[A] \right\}_R F[A, \bar{c}, c] \\ & = \sum_{i=1}^n \int DAD\bar{c}Dc \{ \delta_{f_i} e^{-S[\omega A, \omega \bar{c}, \omega c]} G_i[A] \}_R F[A, \bar{c}, c]. \end{aligned} \quad (170)$$

$G_i[A]$ are enough regular functionals, so

$$\{ \dots G_i[A] \}_R = \{ \dots \}_R G_i[A]. \quad (171)$$

Therefore we have

$$\begin{aligned} & \delta\langle F[A] \rangle \\ &= \sum_{i=1}^n \int DAD\bar{c}Dc \{ \delta_{f_i} e^{iS} [{}^\omega A, {}^\omega \bar{c}, {}^\omega c] \}_R G_i[A] F[A, \bar{c}, c] = 0. \end{aligned} \quad (172)$$

The statement is proved.

To prove the fact that S -matrix is unitary it is enough to prove that S -matrix is gauge independent. And to prove gauge independence of S one can use similar arguments.

11 Conclusion

In this work we have given the generalization of the Connes — Kreimer method in renormalization theory to the case of nonabelian gauge theories. We have introduced the Hopf algebra of diagrams which generalize the corresponding construction of Connes and Kreimer. We have defined a gauge transformation on this Hopf algebra.

We have obtained three main results. The first one is that the gauge transformation is a derivation of the Hopf algebra of diagrams. The second one is that the set of all gauge invariant characters is a group. The third one is that the Riemann — Hilbert problem has a gauge invariant solution if the data of this problem is gauge invariant. We have shown how to simply prove that renormalized S -matrix is gauge invariant.

I would like to thank I.V. Volovich for the problem setup and A.V. Zayakin for very useful discussions.

References

- [1] N.N. Bogoliubov, D.V. Shirkov, Introduction to the theory of quantum fields, Nauka, 1973.
- [2] K. Hepp, Theorie de la renormalisation, New York: Springer-Verlag, 1970.
- [3] Alain Connes, Dirk Kreimer, Renormalization in quantum field theory and Rieman-Hilbert problem I, hep-th/9912092, Commun. Math. Phys. 210: 249-273, 2000.
- [4] Alain Connes, Dirk Kreimer, Renormalization in quantum field theory and Rieman-Hilbert problem II, hep-th/0003188, Commun. Math. Phys. 216: 215-241, 2001.
- [5] Branko Dragovich and Igor Volovich, p -adic strings and noncommutativity, S. Dulij and J. Wess(eds.), Noncommutativity Structures in Mathematics and Physics, 391-399. 2001 Kluwer Academic Publishers. Printed in Netherland.
- [6] I.Ya. Aref'eva and I.V. Volovich, Quantum group particles and non-archimedian geometry, Phys. Lett. B 268 (1991) 179.
- [7] A.A. Bolibruch, Fuchsian differential equations and holomorphic vector bundles, MCCME, Moscow, 2000.

- [8] F. Girelli, T. Krajewski, P. Martinetty, Wave-function renormalization and the Hopf algebra of Connes and Kreimer, math-ph/0103020 v1.
- [9] D.V. Malyshev, Hopf algebra of ribbon graphs and renormalization, hep-th/0112146 v3.
- [10] D. Kastler, Connes-Moscovici-Kreimer Hopf Algebras, math-ph/0104017 v1.
- [11] A.A. Slavnov, L.D. Faddeev, Introduction to the quantum theory of gauge fields, Nauka 1978.
- [12] W. van Suijlekom, Renormalization of gauge fields: a Hopf algebra approach, hep-th/0610137.
- [13] J. Collins, Renormalization, Cambridge monographs in math. physics, Cambridge University Press, 1984. hep-th/0112146 v3.